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A pathwise ergodic theorem for quantum trajectories

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Abstract
If the time evolution of an open quantum system approaches equilibrium in the time mean, then on any single trajectory of any of its unravellings the time-averaged state approaches the same equilibrium state with probability 1. In the case of multiple equilibrium states, the quantum trajectory converges in the mean to a random choice from these states.

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1. Introduction

Stochastic Schrödinger equations and their solutions, quantum trajectories, have been extensively studied in the last 15 years (cf [Car, GaZ]). They provide insight into the behaviour of open quantum systems and they are invaluable for Monte Carlo simulations of the time evolution of such systems, in particular for the numerical determination of equilibrium states.

In performing such simulations one is confronted with the problem of whether it is necessary to average over many trajectories, or if it suffices to calculate the time average over a single trajectory, which is often more convenient (cf [GaZ]).

In this paper, we prove that for any finite-dimensional quantum system and for any initial state the time average of a single quantum trajectory converges to some equilibrium state with probability 1. This result holds true despite the fact that the quantum trajectory itself may stay away from equilibrium forever.

In the simple case, in which there exists only one equilibrium state, the above result implies that the path average converges to this particular state, almost surely and independently of the starting point chosen. So in one sense the quantum trajectory is ergodic in this case: the path average of any observable of the quantum system equals its expectation in the equilibrium state. However, when looked upon as a classical stochastic process with values in the space of all quantum states, the quantum trajectory need not be ergodic, even in this simple and well-behaved case: there may be disjoint regions in the space of all quantum states between which no transitions are possible.

In a previous paper [KüM] we have considered the ergodic properties of the observed output of open quantum systems. We found that quantum systems with finite-dimensional Hilbert spaces and unique equilibrium states lead to ergodic observations. Strangely enough, the techniques needed to prove our present result seem to be entirely different from those used in that paper. Here we make effective use of martingales, which have been introduced to this context in [Bel]. As in [KüM] we concentrate on jump processes in continuous time using the formulation of Davies and Srivinas [Dav, SrD]. But the result also holds for diffusive Schrödinger equations and for quantum evolutions in discrete time, as they occur in repeated measurement situations like the micromaser [WBKM].

The paper is organized as follows. We formulate our result in section 2 and introduce the necessary martingales in section 3. In section 4 the proof of the theorem is given. It is extended to the diffusive and discrete time cases in sections 5 and 6, respectively.

2. The main result

The state of an open quantum system is described by a density matrix ρ on a finite-dimensional Hilbert space $\mathcal{H}$, obeying a master equation $\dot{\rho} = L\rho$, where L is a generator of Lindblad form [Lin],

$$L(\rho) = \mathrm{i}[H, \rho] + \sum_{i=1}^{k} V_i \rho V_i^* - \tfrac{1}{2}(V_i^* V_i \rho + \rho V_i^* V_i).$$

Here $H, V_1, \ldots, V_k$ are linear operators on $\mathcal{H}$, H being self-adjoint.

Conservation of normalization of ρ is expressed by the relation

$$\mathrm{Tr}\, L(\rho) = 0 \quad \text{for all } \rho. \tag{2.1}$$

We consider a decomposition of the generator

$$L = L_0 + \sum_{i=1}^{k} J_i, \tag{2.2}$$

such that $\mathrm{e}^{tL_0}, t \geqslant 0$, and $J_i, i = 1, \ldots, k$, are completely positive. A natural choice is $J_i(\rho) = V_i \rho V_i^*$. Such a decomposition may be interpreted as follows. The open system is under continuous observation by use of k detectors. The reaction of the detectors to the system consists of clicks at random times. The evolution $\rho \mapsto \mathrm{e}^{tL_0}(\rho)$ denotes the change of the state of the system under the condition that during a time interval of length t no clicks are recorded. The operator $\rho \mapsto J_i(\rho)$ on the state space describes the change of state conditioned on the occurrence of a click of detector i.

So, if ρ describes the state of the system at time 0, and if during the time interval $[0, t]$ clicks are recorded at times $t_1, t_2, \ldots, t_n$ of detectors $i_1, i_2, \ldots, i_n$ respectively, and no more, then, up to normalization, the state at time t is given by

$$\vartheta_t((t_1, i_1), \ldots, (t_n, i_n)) = \mathrm{e}^{(t-t_n)L_0} J_{i_n} \mathrm{e}^{(t_n - t_{n-1})L_0} \cdots \mathrm{e}^{(t_2 - t_1)L_0} J_{i_1} \mathrm{e}^{t_1 L_0}(\rho). \tag{2.3}$$

The probability density for these clicks to occur is equal to the trace of ϑ_t in (2.3). We shall denote the normalized density matrix $\vartheta_t/\mathrm{Tr}(\vartheta_t)$ by Θ_t.

We imagine the experiment to continue indefinitely. The observation process will then produce an infinite detection record $((t_1, i_1), (t_2, i_2), (t_3, i_3), \ldots)$, where we assume that

$0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant \cdots$, and $\lim_{n\to\infty} t_n = \infty$ (i.e., the clicks do not accumulate). Let Ω denote the space of all such detection records with Lebesgue measure

$$\mathrm{d}\omega = \sum_{n=0}^{\infty} \sum_{i_1=1}^{k} \cdots \sum_{i_n=1}^{k} \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

As was shown in [KüM], each initial state ϑ_0 determines a probability measure $\mathbb{P}^{\vartheta_0}$ on Ω whose restriction to the time interval $[0, t]$ has density $\mathrm{Tr}(\vartheta_t)$ as described above. We may consider $(\Theta_t)_{t\geqslant 0}$ as a stochastic process on this probability space taking values in the density matrices. A path of this process is called a *quantum trajectory*. We thus obtain an unravelling of the state at time $t \geqslant 0$:

$$T_t(\vartheta_0) := \mathrm{e}^{tL}(\vartheta_0) = \int_\Omega \Theta_t(\omega)\mathbb{P}^{\vartheta_0}(\mathrm{d}\omega) = \mathbb{E}^{\vartheta_0}(\Theta_t). \tag{2.4}$$

So far the framework is essentially the same as described in our previous paper [KüM]. It is the framework frequently used in computer simulations (cf, e.g., [Car, GaZ]). If one is only interested in the average evolution e^{tL}, then the decomposition (2.2) can be chosen at will.

We now address the question, what can be said about the asymptotic behaviour of each single quantum trajectory $(\Theta_t(\omega))_{t\geqslant 0}$.

Let us denote by $\mathcal{E}$ the space of equilibrium states, i.e. density matrices ρ which are left invariant by the average evolution T_t. Since the Hilbert space is finite dimensional the limit

$$P(\vartheta) = \lim_{t\to\infty} \frac{1}{t} \int_0^t T_s(\vartheta)\,\mathrm{d}s \tag{2.5}$$

exists and projects any density matrix ϑ onto the space $\mathcal{E}$ of equilibrium states.

Theorem 1. *Suppose that $T_t = \mathrm{e}^{tL}$ has only a single equilibrium state ρ. Then for every initial state ϑ_0 the quantum trajectory $(\Theta_t)_{t\geqslant 0}$ satisfies*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Theta_s(\omega)\,\mathrm{d}s = \rho,$$

for almost all ω with respect to the probability measure $\mathbb{P}^{\vartheta_0}$.

More generally, in the case where there is more than one equilibrium state, one has almost surely

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Theta_s(\omega)\,\mathrm{d}s = \Theta_\infty(\omega),$$

where Θ_∞ is a random variable, depending on the initial state ϑ_0, and taking values in the equilibrium states. The expectation of Θ_∞ is $P(\vartheta_0)$.

The proof of this theorem is inspired by the arguments leading to Breiman's strong law of large numbers for Markov chains [Bre] (see also [Kre]), which however does not apply directly to the situation of continuous time quantum trajectories. Our proof, based on the martingale convergence theorem (section 3), will be given in section 4. In our discussion we make free use of standard stochastic notation and arguments for which we refer, e.g., to [Doo, vKa, ChW].

3. Martingales

The process $(\Theta_t)_{t\geqslant 0}$ consists of smooth evolution according to e^{tL_0} interrupted by jumps of different types $i = 1, \ldots, k$, namely $\Theta_t \mapsto J_i\Theta_t/\mathrm{Tr}(J_i\Theta_t)$. Let $N_i(t)$ denote the number of jumps of type i before time t. In the theory of point processes [Ram, vKa, Bar] it is well known that, from the probability density (2.3), it follows that the unconditioned probability density of the occurrence of a jump of type i at time u, given the state Θ_s at time s is

$$\mathrm{Tr}(T_{t-u} J_i T_{u-s}(\Theta_s)) = \mathrm{Tr}(J_i T_{u-s}(\Theta_s))$$

for $t \geqslant u \geqslant s \geqslant 0$, independent of $t \geqslant u$. Let $\mathbb{E}_s^{\vartheta_0}$ denote expectation with respect to $\mathbb{P}^{\vartheta_0}$, given the process up to time s.

In a similar way as (2.4) follows from (2.3), it is easy to show that $T_{u-s}(\Theta_s) = \mathbb{E}_s^{\vartheta_0}(\Theta_u)$, and therefore

$$\mathbb{E}_s^{\vartheta_0}\big(N_t^i - N_s^i\big) = \int_s^t \mathrm{Tr}(T_{t-u} J_i T_{u-s}(\Theta_s))\,\mathrm{d}u = \mathbb{E}_s^{\vartheta_0}\left(\int_s^t \mathrm{Tr}(J_i(\Theta_u))\,\mathrm{d}u\right).$$

If we now denote by $\tilde{N}_t^i$ the process

$$\tilde{N}_t^i := N_t^i - \int_0^t \mathrm{Tr}\, J_i(\Theta_u)\,\mathrm{d}u,$$

then $\tilde{N}_t^i$ is a *martingale* [Doo], i.e., for all $0 \leqslant s \leqslant t$,

$$\mathbb{E}_s^{\vartheta_0}\big(\tilde{N}_t^i\big) = \tilde{N}_s^i.$$

$\big(\tilde{N}_t^i\big)_{t\geqslant 0}$ is the *compensated number process* of jumps of type i.

Lemma 2. *The quantum trajectory $(\Theta_t)_{t\geqslant 0}$ satisfies the stochastic Schrödinger equation [Car, BGM]*

$$\mathrm{d}\Theta_t = L(\Theta_t)\,\mathrm{d}t + \sum_{i=1}^k \left(\frac{J_i(\Theta_t)}{\mathrm{Tr}(J_i(\Theta_t))} - \Theta_t\right)\mathrm{d}\tilde{N}_t^i, \tag{3.1}$$

where the stochastic differential equation is interpreted in the sense of Itô [ChW].

Proof. Between jumps ϑ_t evolves according to $\frac{\mathrm{d}}{\mathrm{d}t}\vartheta_t = L_0(\vartheta_t)$, at a jump of type i it jumps from ϑ_t to $J_i(\vartheta_t)$. It follows that the normalized state $\Theta_t = \vartheta_t/\mathrm{Tr}(\vartheta_t)$ satisfies

$$\mathrm{d}\Theta_t = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vartheta_t}{\mathrm{Tr}(\vartheta_t)}\right)\mathrm{d}t + \sum_{i=1}^k \left(\frac{J_i(\Theta_t)}{\mathrm{Tr}(J_i(\Theta_t))} - \Theta_t\right)\mathrm{d}N_t^i.$$

Since between jumps we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vartheta_t}{\mathrm{Tr}(\vartheta_t)}\right) = \frac{L_0(\vartheta_t)}{\mathrm{Tr}(\vartheta_t)} - \frac{\vartheta_t \cdot \mathrm{Tr}(L_0(\vartheta_t))}{\mathrm{Tr}(\vartheta_t)^2} = L_0(\Theta_t) - \Theta_t \cdot \mathrm{Tr}(L_0(\Theta_t))$$

and since $\mathrm{d}N_t^i = \mathrm{d}\tilde{N}_t^i + \mathrm{Tr}(J_i(\Theta_t))\,\mathrm{d}t$, we have, using that $\mathrm{Tr} \circ L = 0$,

$$\begin{aligned}\mathrm{d}\Theta_t &= (L_0(\Theta_t) - \Theta_t \cdot \mathrm{Tr}\, L_0(\Theta_t))\,\mathrm{d}t + \sum_{i=1}^k \left(\frac{J_i(\Theta_t)}{\mathrm{Tr}(J_i(\Theta_t))} - \Theta_t\right) \cdot \big(\mathrm{d}\tilde{N}_t^i + \mathrm{Tr}(J_i(\Theta_t))\,\mathrm{d}t\big)\\ &= \left(L_0 + \sum_{i=1}^k J_i\right)(\Theta_t)\,\mathrm{d}t - \Theta_t \cdot \mathrm{Tr}\left(\left(L_0 + \sum_{i=1}^k J_i\right)(\Theta_t)\right)\mathrm{d}t\\ &\quad + \sum_{i=1}^k \left(\frac{J_i(\Theta_t)}{\mathrm{Tr}(J_i(\Theta_t))} - \Theta_t\right) \cdot \mathrm{d}\tilde{N}_t^i = L(\Theta_t)\,\mathrm{d}t + \sum_{i=1}^k \left(\frac{J_i(\Theta_t)}{\mathrm{Tr}(J_i(\Theta_t))} - \Theta_t\right)\mathrm{d}\tilde{N}_t^i.\end{aligned}$$

□

The process Θ_t starts at $\Theta_0 = \vartheta_0$. Let us now consider two other stochastic processes

$$M_t := \Theta_t - \vartheta_0 - \int_0^t L(\Theta_s)\,\mathrm{d}s = \int_0^t \sum_{i=1}^k \left(\frac{J_i(\Theta_s)}{\mathrm{Tr}\, J_i(\Theta_s)} - \Theta_s\right)\mathrm{d}\tilde{N}_s^i \qquad (t \geqslant 0),$$

and

$$Y_t := \int_1^t \frac{1}{s} \sum_{i=1}^{k} \left(\frac{J_i(\Theta_s)}{\mathrm{Tr}\, J_i(\Theta_s)} - \Theta_s \right) \mathrm{d}\tilde{N}_s^i = \int_1^t \frac{1}{s}\, \mathrm{d}M_s \qquad (t \geqslant 1).$$

From the fact that $\widetilde{N}_t^i$ is a martingale, it follows that these processes are martingales as well [ChW]. We now come to the main result of this section.

Proposition 3. *For any initial state ϑ_0 the quantum trajectory $(\Theta_t(\omega))_{t\geqslant 0}$ satisfies*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t L(\Theta_s(\omega))\, \mathrm{d}s = 0$$

almost surely with respect to $\mathbb{P}^{\vartheta_0}$.

Proof. Let us first consider the martingale Y_t which takes values in the self-adjoint matrices. In order to conclude from the martingale convergence theorem [Doo] that $(Y_t)_{t\geqslant 0}$ converges almost surely, we show that $\mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\,\big(Y_t^2\big)\big)$ remains bounded.

Denote the coefficient $\left(\frac{J_i(\Theta_t)}{\mathrm{Tr}\, J_i(\Theta_t)} - \Theta_t\right)$ by X_t^i. Then

$$\mathrm{d}Y_t = \sum_{i=1}^{k} \frac{1}{t} X_t^i\, \mathrm{d}\widetilde{N}_t^i.$$

By the Itô rules for jump processes [ChW] $\mathrm{d}\tilde{N}_t^i\, \mathrm{d}\tilde{N}_t^j = \mathrm{d}N_t^i\, \mathrm{d}N_t^j = \delta_{ij}\, \mathrm{d}N_t^i$, we find that

$$(\mathrm{d}Y_t)^2 = \frac{1}{t^2} \sum_{i=1}^{k} \sum_{j=1}^{k} X_t^i X_t^j\, \mathrm{d}\tilde{N}_t^i\, \mathrm{d}\tilde{N}_t^j = \frac{1}{t^2} \sum_{i=1}^{k} \big(X_t^i\big)^2\, \mathrm{d}N_t^i.$$

From $\mathrm{d}\big(Y_t^2\big) = 2Y_t\, \mathrm{d}Y_t + (\mathrm{d}Y_t)^2$ and $\mathbb{E}^{\vartheta_0}\big(\mathrm{d}\tilde{N}_t^i\big) = 0$, hence $\mathbb{E}^{\vartheta_0}(\mathrm{Tr}(Y_t\, \mathrm{d}Y_t)) = 0$, we obtain $\mathbb{E}^{\vartheta_0}\big(\mathrm{d}\big(\mathrm{Tr}\big(Y_t^2\big)\big)\big) = \mathbb{E}^{\vartheta_0}(\mathrm{Tr}((\mathrm{d}Y_t)^2))$. Therefore, since $\mathbb{E}^{\theta_0}\big(\mathrm{d}N_t^i\big) = \mathbb{E}^{\theta_0}(\mathrm{Tr}\, J_i(\Theta_t))\, \mathrm{d}t$,

$$\mathrm{d}\mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\,\big(Y_t^2\big)\big) = \mathbb{E}^{\vartheta_0}(\mathrm{Tr}((\mathrm{d}Y_t)^2)) = \frac{1}{t^2} \sum_{i=1}^{k} \mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\big(\big(X_t^i\big)^2\big) \cdot \mathrm{Tr}(J_i(\Theta_t))\big)\, \mathrm{d}t$$

hence

$$\mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\big(Y_t^2\big)\big) = \int_1^t \frac{1}{s^2} \sum_{i=1}^{k} \mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\big(\big(X_s^i\big)^2\big) \cdot \mathrm{Tr}\, J_i(\Theta_s)\big)\, \mathrm{d}s \leqslant 4 \sum_{i=1}^{k} \|J_i\|.$$

In this sense, $(Y_t)_{t\geqslant 1}$ is L^2-bounded and it follows that Y_t converges almost surely to some random variable Y. In particular, since Y_t is continuous up to finitely many jumps on compact time intervals and has a limit as $t \to \infty$ almost surely, it is bounded almost surely. Therefore, applying the partial integration formula, which is also valid if Y_t has jumps, we obtain for $t \geqslant 1$

$$M_t = M_1 + \int_1^t s\, \mathrm{d}Y_s = M_1 + sY_s|_1^t - \int_1^t Y_s\, \mathrm{d}s = M_1 + tY_t - \int_1^t Y_s\, \mathrm{d}s,$$

therefore,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} M_t &= \lim_{t\to\infty} \frac{1}{t} M_1 + \lim_{t\to\infty} Y_t - \lim_{t\to\infty} \frac{1}{t} \int_1^t Y_s\, \mathrm{d}s \\ &= 0 + Y - Y \\ &= 0. \end{aligned}$$

We thus conclude that

$$\lim_{t\to\infty} \frac{1}{t} \left(\Theta_t - \vartheta_0 - \int_0^t L(\Theta_s)\, \mathrm{d}s \right) = 0.$$

As $(\Theta_t - \vartheta_0)$ remains bounded, the statement of the proposition follows. □

4. Proof of the main result

We shall prove theorem 1 in two steps.

Step 1. If P is given as in (2.5), then for any initial state ϑ_0 the limit

$$\lim_{t\to\infty} P(\Theta_t) =: \Theta_\infty$$

exists almost surely with respect to $\mathbb{P}^{\vartheta_0}$, and satisfies $\mathbb{E}^{\vartheta_0}(\Theta_\infty) = P(\vartheta_0)$.

Proof. Acting with the operator P on both sides of (3.1) in lemma 2 we see that $\mathbb{E}^{\vartheta_0}(P(\mathrm{d}\Theta_t)) = 0$, hence $(P(\Theta_t))_{t\geqslant 0}$ is a martingale. Since it takes values in the states it is bounded, and therefore it converges almost surely, say to the random variable Θ_∞. The expectation of Θ_∞ is $P(\vartheta_0)$, the initial value of the martingale $(P(\Theta_t))_{t\geqslant 0}$. □

Step 2. For any initial state ϑ_0 we have, almost surely with respect to $\mathbb{P}^{\vartheta_0}$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (\Theta_u - P(\Theta_u))\,\mathrm{d}u = 0. \tag{4.1}$$

Proof. First we show that, for all $s \geqslant 0$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (\Theta_u - T_s(\Theta_u))\,\mathrm{d}u = 0. \tag{4.2}$$

Indeed, since $\frac{\mathrm{d}}{\mathrm{d}v}T_v = T_v L$:

$$\int_0^t (T_s - \mathrm{id})(\Theta_u)\,\mathrm{d}u = \int_0^t\int_0^s T_v L(\Theta_u)\,\mathrm{d}v\,\mathrm{d}u = \int_0^s T_v\left(\int_0^t L(\Theta_u)\,\mathrm{d}u\right)\mathrm{d}v.$$

Dividing by t and taking the limit $t\to\infty$, we obtain (4.2) by proposition 3.

Clearly, averaging (4.2) over $[0, s]$ preserves its validity:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \left(\Theta_u - \frac{1}{s}\int_0^s T_v(\Theta_u)\,\mathrm{d}v\right)\mathrm{d}u = 0.$$

In the above we want to take the limit $s\to\infty$ *before the limit* $t\to\infty$, in order to obtain the statement (4.1) to be proved.

This is allowed since $\mathcal{H}$ is finite dimensional. Then for $\varepsilon > 0$ there exists $s > 0$ such that $\left\|\frac{1}{s}\int_0^s T_v\,\mathrm{d}v - P\right\| < \frac{\varepsilon}{2}$, hence

$$\left\|\frac{1}{s}\int_0^s T_v(\Theta_u(\omega))\,\mathrm{d}v - P(\Theta_u(\omega))\right\| < \frac{\varepsilon}{2},$$

uniformly in u. For $\mathbb{P}^{\vartheta_0}$, almost every $\omega\in\Omega$, we find t_0 such that for $t > t_0$

$$\left\|\frac{1}{t}\int_0^t\left(\Theta_u(\omega) - \frac{1}{s}\int_0^s T_v(\Theta_u(\omega))\,\mathrm{d}v\right)\mathrm{d}u\right\| < \frac{\varepsilon}{2}.$$

Then, we obtain for such t

$$\begin{aligned}
&\left\|\frac{1}{t}\int_0^t (\Theta_u(\omega) - P(\Theta_u(\omega)))\,\mathrm{d}u\right\| \\
&\quad= \left\|\frac{1}{t}\int_0^t\left(\Theta_u(\omega) - P(\Theta_u(\omega)) + \frac{1}{s}\int_0^s T_v(\Theta_u(\omega))\,\mathrm{d}v\right.\right. \\
&\qquad\left.\left. - \frac{1}{s}\int_0^s T_v(\Theta_u(\omega))\,\mathrm{d}v\right)\mathrm{d}u\right\| \leqslant \left\|\frac{1}{t}\int_0^t\left(\Theta_u(\omega) - \frac{1}{s}\int_0^s T_v(\Theta_u(\omega))\,\mathrm{d}v\right)\mathrm{d}u\right\| \\
&\qquad+ \left\|\frac{1}{t}\int_0^t\left(\frac{1}{s}\int_0^s T_v(\Theta_u(\omega))\,\mathrm{d}v - P(\Theta_u(\omega))\right)\mathrm{d}u\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

5. Diffusive quantum trajectories

The ergodic result obtained above is not confined to jump processes. Solutions of the master equation $\dot{\rho} = L\rho$ with

$$L(\rho) = \mathrm{i}[H, \rho] + \sum_{j=1}^{k} V_j \rho V_j^* - \frac{1}{2}(V_j^* V_j \rho + \rho V_j^* V_j)$$

can alternatively be unravelled into a diffusion Θ_t on the state space, satisfying the stochastic differential equation [Bel, Car, BGM],

$$\mathrm{d}\Theta_t = L(\Theta_t)\,\mathrm{d}t + \sum_{i=1}^{k} X_t^i\,\mathrm{d}\widetilde{W}_t^i,$$

where

$$X_t^i = \Theta_t V_i^* + V_i \Theta_t - \mathrm{Tr}(\Theta_t V_i^* + V_i \Theta_t) \cdot \Theta_t$$

and

$$\mathrm{d}\widetilde{W}_t^i = \mathrm{d}W_t^i - \mathrm{Tr}(\Theta_t V_i^* + V_i \Theta_t)\,\mathrm{d}t.$$

As usual, $W_t^i, i = 1, \ldots, k$, denote pairwise independent real-valued Wiener processes. Such 'state diffusions' arise, for instance, in homodyne detection of the field strength of fluorescence light [Car]. In this situation our main theorem takes the following form.

Theorem 4. *We have almost surely*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Theta_s(\omega)\,\mathrm{d}s = \Theta_\infty(\omega),$$

where Θ_∞ is a random variable, depending on the initial state ϑ_0 and taking values in the equilibrium states. Again, the expectation of Θ_∞ is $P\vartheta_0$.

Proof. We follow the same line of argument as for jump processes. Here we only discuss the modifications needed for the diffusive case. We consider the stochastic processes $(M_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 1}$ given by

$$M_t := \Theta_t - \vartheta_0 - \int_0^t L(\Theta_s)\,\mathrm{d}s = \int_0^t \sum_{i=1}^{k} X_s^i\,\mathrm{d}\tilde{W}_s^i,$$

and

$$Y_t := \int_1^t \frac{1}{s} \sum_{i=1}^{k} X_s^i\,\mathrm{d}\tilde{N}_s^i = \int_1^t \frac{1}{s}\,\mathrm{d}M_s.$$

As was shown in [Bel, BGM], these are martingales. Again $\mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\big(Y_t^2\big)\big)$ remains bounded. Indeed $\mathrm{d}\widetilde{W}_t^i\,\mathrm{d}\widetilde{W}_t^j = \mathrm{d}W_t^i\,\mathrm{d}W_t^j = \mathrm{d}t$ by the Itô rules and $\mathbb{E}^{\vartheta_0}\big(\mathrm{d}\big(\mathrm{Tr}\,Y_t^2\big)\big) = \mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\big(\mathrm{d}Y_t\big)^2\big)$ with

$$\big(\mathrm{d}Y_t\big)^2 = \frac{1}{t^2} \sum_{i=1}^{k} \sum_{j=1}^{k} X_t^i X_t^j\,\mathrm{d}\tilde{W}_t^i\,\mathrm{d}\tilde{W}_t^j = \frac{1}{t^2} \sum_{i=1}^{k} \big(X_t^i\big)^2\,\mathrm{d}t,$$

so that

$$\mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\,Y_t^2\big) = \int_1^t \frac{1}{s^2} \sum_{i=1}^{k} \mathbb{E}^{\vartheta_0}\big(\mathrm{Tr}\,\big(X_s^i\big)^2\big)\,\mathrm{d}s \leqslant 4 \sum_{i=1}^{k} \|V_i\|^2.$$

The partial integration argument, which is also valid for diffusions, leads to proposition 3. Steps 1 and 2 in the proof of the main result remain unchanged. □

6. Quantum trajectories in discrete time

Our ergodic theorem also has a natural version in discrete time. Let us briefly sketch the setting. A time evolution in discrete time is given by the powers of a completely positive operator T with $\mathrm{Tr} \circ T = \mathrm{Tr}$. A Kraus decomposition

$$T(\rho) = \sum_{i=1}^{k} V_i \rho V_i^*$$

of T leads to an unravelling of this time evolution. Let Ω be the set of all infinite sequences $(\omega_1, \omega_2, \ldots)$ with $\omega_j = 1, \ldots, k$. An initial state ϑ_0 induces a probability measure $\mathbb{P}^{\vartheta_0}$ on Ω which is uniquely determined by the condition

$$\mathbb{P}^{\vartheta_0}(\{\omega \in \Omega : \omega_1 = i_1, \omega_2 = i_2, \ldots, \omega_n = i_n\}) = \mathrm{Tr}\left(V_{i_n} \cdots V_{i_1} \vartheta_0 V_{i_1}^* \cdots V_{i_n}^*\right).$$

Then an unravelling of the time evolution $(T^n)_{n\geqslant 0}$ is given by the Markov chain $(\Theta_n)_{n\geqslant 0}$ on $(\Omega, \mathbb{P}^{\vartheta_0})$ with

$$\Theta_n(\omega) = \frac{V_{i_n} \cdots V_{i_1} \vartheta_0 V_{i_1}^* \cdots V_{i_n}^*}{\mathrm{Tr}\left(V_{i_n} \cdots V_{i_1} \vartheta_0 V_{i_1}^* \cdots V_{i_n}^*\right)}.$$

Theorem 5. *As $N \to \infty$, the averaged process*

$$\frac{1}{N} \sum_{n=0}^{N-1} \Theta_n(\omega)$$

converges $\mathbb{P}^{\vartheta_0}$ almost surely to a random equilibrium state Θ_∞ with expectation $P(\vartheta_0)$.

The proof is a discrete version of the argument in the previous sections, which corresponds to a variation on Breiman's individual ergodic theorem for Markov chains [Bre, Kre].

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